Test for PreAlgebra

MO

Regular dash
$\overline{x}$

bold dash
x

subtraction dash
$\overline{x}$

en dash
$\bar{x}$

em dash
x

overbar
x

TEXT

Regular dash
$\bar{x}$

bold dash
x

subtraction dash
$\overline{x}$

en dash
$\bar{x}$

em dash

x

overbar

$\bar{x}$

Fancy E Test

- Graph exponential functions.
- Graph exponential functions using transformations.

U+0190 Ɛ Latin Capital Letter Open E
MathML $\mathcal{E}$

Inline Ɛ

Ɛ ɛ U+025B
MathMl $\mathcal{E}$

Inline Ɛ

ξ Small Xi Letter U+03BE
MathMl

Inline ξ

Derived copy of Real Numbers: Algebra Essentials
In this section students will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In

this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, ...\}$ where the ellipsis (…) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

Equation:

$$\overset{\text{negative integers}}{\ldots, -3, -2, -1,} \quad \overset{\text{zero}}{0,} \quad \overset{\text{positive integers}}{1, 2, 3, \cdots}$$

The set of **rational numbers** is written as $\left\{\frac{m}{n} \;\; m \text{ and } n \text{ are integers and } n \neq 0\right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8} = 1.875$,or

2. a repeating decimal: $\frac{4}{11} = 0.36363636\ldots = 0.36$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example:
Exercise:

Problem:
Writing Integers as Rational Numbers

Write each of the following as a rational number.

a. 7
b. 0
c. –8

Solution:

Write a fraction with the integer in the numerator and 1 in the denominator.

a. $7 = \frac{7}{1}$
b. $0 = \frac{0}{1}$
c. $-8 = -\frac{8}{1}$

Note:
Exercise:

Problem: Write each of the following as a rational number.

a. 11

b. 3
c. –4

Solution:

a. $\frac{11}{1}$
b. $\frac{3}{1}$
c. $-\frac{4}{1}$

Example:
Exercise:

Problem:
Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$
c. $\frac{13}{25}$

Solution:

Write each fraction as a decimal by dividing the numerator by the denominator.

a. $-\frac{5}{7} = -0.714285$,a repeating decimal
b. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal
c. $\frac{13}{25} = 0.52$, a terminating decimal

tumTEST -0.714285 MORE TEST -0.714285 EVEN MORE TEST -0.714285 test test test 714285 test test tests pwm math -0.714285

. in op/mover in row. -0.714285
. in op everything in a row. -0.714285
NOTHING IN A ROW -0.714285
everything but mover in a row -0.714285
EM DASH -0.714285
UNDERLINE -0.714285
EACH NUMBER GETS IT'S OWN OVERBAR -0.714285

DASH -0.714285
overbar in text form $-0.7\overline{14285}$
emdash things in text $-0.7\overline{14285}$
EACH NUMBER GETS IT'S OWN DASH $-0.\overline{714285}$

EACH NUMBER GETS IT'S OWN UNDERLINE $-0.\overline{714285}$

Note:
Exercise:

Problem:

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$
b. $\frac{8}{13}$
c. $-\frac{17}{20}$

Solution:

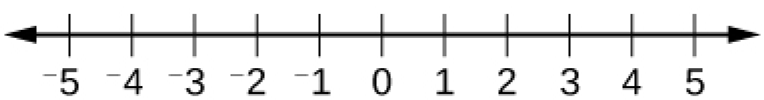
a. 4 (or 4.0), terminating;

b. 0.615384,repeating;
c. –0.85, terminating

Real Numbers

Given any number *n*, we know that *n* is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in [link]**.**



The real number line

Example:

Exercise:

Problem:
Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution:

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

Note:
Exercise:

Problem:

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$
b. $-11.411411411\ldots$
c. $\frac{47}{19}$
d. $-\frac{\sqrt{5}}{2}$
e. 6.210735

Solution:

a. positive, irrational; right
b. negative, rational; left
c. positive, rational; right
d. negative, irrational; left
e. positive, rational; right

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See [link] and [link].
- Determine whether a number is rational or irrational by writing it as a decimal. See [link].
- The rational numbers and irrational numbers make up the set of real numbers. See [link]. A number can be classified as natural, whole, integer, rational, or irrational. See [link].
- The order of operations is used to evaluate expressions. See [link].

- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See [link].
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See [link]. They take on a numerical value when evaluated by replacing variables with constants. See [link], [link], and [link]
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See [link] and [link].

Verbal

Exercise:

Problem:

Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

Solution:

irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational.

Exercise:

Problem:

What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

Exercise:

Problem:

What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

Solution:

The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. This is because the same operation is performed (either addition or subtraction), so the terms can be re-ordered.

Numeric

For the following exercises, simplify the given expression.

Exercise:

Problem: $10+2\ \times\ (5-3)$

Exercise:

Problem: $6\div 2-\left(81\div 3^2\right)$

Solution:

-6

Exercise:

Problem: $18+(6-8)^3$

Exercise:

Problem: $-2\ \times\ \left[16\div (8-4)^2\right]^2$

Solution:

-2

Exercise:

Problem:$4-6+2\ \times\ 7$

Exercise:

Problem:$3\left(5-8\right)$

Solution:

-9

Exercise:

Problem:$4+6-10\div 2$

Exercise:

Problem:$12\div\left(36\div 9\right)+6$

Solution:

9

Exercise:

Problem:$\left(4+5\right)^{2}\div 3$

Exercise:

Problem:$3-12\ \times\ 2+19$

Solution:

4

Exercise:

Problem: $2 + 8 \times 7 \div 4$

Exercise:

Problem: $5 + (6 + 4) - 11$

Solution:

4

Exercise:

Problem: $9 - 18 \div 3^2$

Exercise:

Problem: $14 \times 3 \div 7 - 6$

Solution:

0

Exercise:

Problem: $9 - (3 + 11) \times 2$

Exercise:

Problem: $6 + 2 \times 2 - 1$

Solution:

9

Exercise:

Problem: $64 \div (8 + 4 \times 2)$

Exercise:

Problem: $9 + 4\left(2^2\right)$

Solution:

25

Exercise:

Problem: $(12 \div 3 \times 3)^2$

Exercise:

Problem: $25 \div 5^2 - 7$

Solution:

-6

Exercise:

Problem: $(15 - 7) \times (3 - 7)$

Exercise:

Problem: $2 \times 4 - 9(-1)$

Solution:

17

Exercise:

Problem: $4^2 - 25 \times \frac{1}{5}$

Exercise:

Problem:$12\left(3-1\right)\div 6$

Solution:

4

Algebraic

For the following exercises, solve for the variable.

Exercise:

Problem:$8\left(x+3\right)=64$

Exercise:

Problem:$4y+8=2y$

Solution:

-4

Exercise:

Problem:$\left(11a+3\right)-18a=-4$

Exercise:

Problem:$4z-2z\left(1+4\right)=36$

Solution:

-6

Exercise:

Problem:$4y(7-2)^2=-200$

Exercise:

Problem:$-(2x)^2+1=-3$

Solution:

±1

Exercise:

Problem:$8\left(2+4\right)-15b=b$

Exercise:

Problem:$2\left(11c-4\right)=36$

Solution:

2

Exercise:

Problem:$4\left(3-1\right)x=4$

Exercise:

Problem:$\frac{1}{4}\left(8w-4^2\right)=0$

Solution:

2

For the following exercises, simplify the expression.
Exercise:

Problem: $4x + x\left(13 - 7\right)$

Exercise:

Problem: $2y - \left(4\right)^2 y - 11$

Solution:

$-14y - 11$

Exercise:

Problem: $\frac{a}{2^3}\left(64\right) - 12a \div 6$

Exercise:

Problem: $8b - 4b\left(3\right) + 1$

Solution:

$-4b + 1$

Exercise:

Problem: $5l \div 3l \times \left(9 - 6\right)$

Exercise:

Problem: $7z - 3 + z \times 6^2$

Solution:

$43z - 3$

Exercise:

Problem: $4 \times 3 + 18x \div 9 - 12$

Exercise:

Problem: $9\left(y+8\right)-27$

Solution:

$9y+45$

Exercise:

Problem: $\left(\frac{9}{6}t-4\right)2$

Exercise:

Problem: $6+12b-3\;\times\;6b$

Solution:

$-6b+6$

Exercise:

Problem: $18y-2\left(1+7y\right)$

Exercise:

Problem: $\left(\frac{4}{9}\right)^{2}\;\times\;27x$

Solution:

$\frac{16x}{3}$

Exercise:

Problem: $8\left(3-m\right)+1\left(-8\right)$

Exercise:

Problem:$9x+4x\left(2+3\right)-4\left(2x+3x\right)$

Solution:

$9x$

Exercise:

Problem:$5^2-4\left(3x\right)$

Real-World Applications

For the following exercises, consider this scenario: Fred earns $40 mowing lawns. He spends $10 on mp3s, puts half of what is left in a savings account, and gets another $5 for washing his neighbor’s car.

Exercise:

Problem:

Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

Solution:

$\frac{1}{2}\left(40-10\right)+5$

Exercise:

Problem: How much money does Fred keep?

For the following exercises, solve the given problem.

Exercise:

Problem:

According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

Solution:

irrational number

Exercise:

Problem:

Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

Exercise:

Problem: Write the equation that describes the situation.

Solution:

$g + 400 - 2\left(600\right) = 1200$

Exercise:

Problem: Solve for g.

For the following exercise, solve the given problem.

Exercise:

Problem:

Ramon runs the marketing department at his company. His department gets a budget every year, and every year, he must spend the entire budget without going over. If he spends less than the budget, then his department gets a smaller budget the following year. At the beginning of this year, Ramon got $2.5 million for the annual marketing budget. He must spend the budget such that $2,500,000 - x = 0$. What property of addition tells us what the value of *x* must be?

Solution:

inverse property of addition

Technology

For the following exercises, use a graphing calculator to solve for *x*. Round the answers to the nearest hundredth.

Exercise:

Problem: $0.5(12.3)^2 - 48x = \frac{3}{5}$

Exercise:

Problem: $(0.25 - 0.75)^2x - 7.2 = 9.9$

Solution:

68.4

Extensions

Exercise:

Problem:

If a whole number is not a natural number, what must the number be?

Exercise:

Problem:

Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

Solution:

true

Exercise:

Problem:

Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

Exercise:

Problem:

Determine whether the simplified expression is rational or irrational: $\sqrt{-18-4\left(5\right)\left(-1\right)}$.

Solution:

irrational

Exercise:

Problem:

Determine whether the simplified expression is rational or irrational: $\sqrt{-16+4\left(5\right)+5}$.

Exercise:

Problem:

The division of two whole numbers will always result in what type of number?

Solution:

rational

Exercise:

Problem:

What property of real numbers would simplify the following expression: $4+7(x-1)$?

Glossary

algebraic expression
: constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
: the sum of three numbers may be grouped differently without affecting the result; in symbols, $a+(b+c)=(a+b)+c$

associative property of multiplication
: the product of three numbers may be grouped differently without affecting the result; in symbols, $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

base
: in exponential notation, the expression that is being multiplied

commutative property of addition
: two numbers may be added in either order without affecting the result; in symbols, $a+b=b+a$

commutative property of multiplication

two numbers may be multiplied in any order without affecting the result; in symbols, $a \cdot b = b \cdot a$

constant
: a quantity that does not change value

distributive property
: the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a \cdot (b + c) = a \cdot b + a \cdot c$

equation
: a mathematical statement indicating that two expressions are equal

exponent
: in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
: a shorthand method of writing products of the same factor

formula
: an equation expressing a relationship between constant and variable quantities

identity property of addition
: there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a + 0 = a$

identity property of multiplication
: there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1 = a$

integers
: the set consisting of the natural numbers, their opposites, and 0: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

inverse property of addition

for every real number a,there is a unique number, called the additive inverse (or opposite), denoted $-a$,which, when added to the original number, results in the additive identity, 0; in symbols, $a+(-a)=0$

inverse property of multiplication

for every non-zero real number a,there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$,which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a \cdot \frac{1}{a}=1$

irrational numbers

the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers

the set of counting numbers: $\{1, 2, 3 ,...\}$

order of operations

a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers

the set of all numbers of the form $\frac{m}{n}$,where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line

a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers

the sets of rational numbers and irrational numbers taken together

variable

a quantity that may change value

whole numbers

the set consisting of 0 plus the natural numbers: $\{0, 1, 2, 3\ , \ldots\}$

Derived copy of Systems of Linear Equations: Three Variables
In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

(credit: “Elembis,” Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at

a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

Equation:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

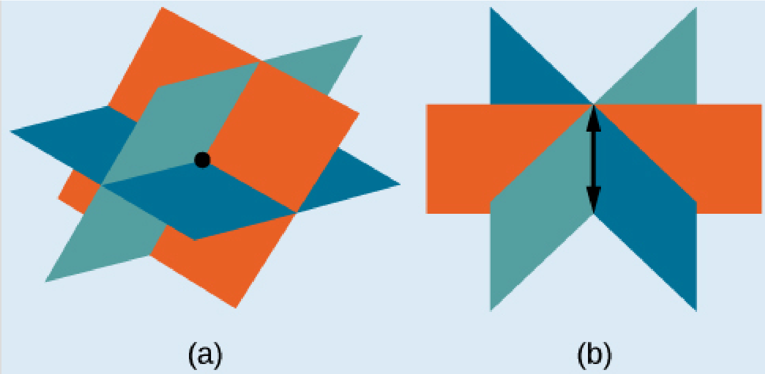
The **solution set** to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Note:

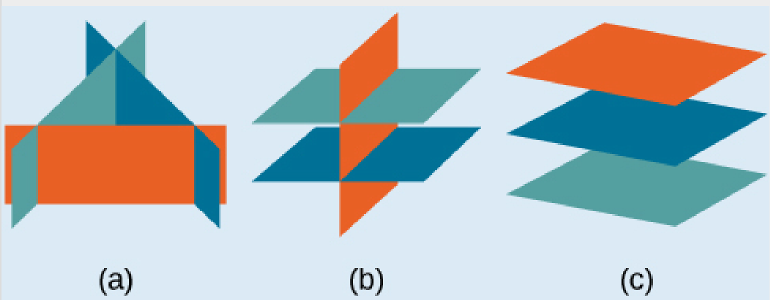
Number of Possible Solutions

[link] and [link] illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.



(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example:
Exercise:

Problem:
Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.
Equation:

$$
\begin{aligned}
x + y + z &= 2 \\
6x - 4y + 5z &= 31 \\
5x + 2y + 2z &= 13
\end{aligned}
$$

Solution:

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$ $(3) + (-2) + (1) = 2$ True	$6x - 4y + 5z = 31$ $6(3) - 4(-2) + 5(1) = 31$ $18 + 8 + 5 = 31$ True	$5x + 2y + 2z = 13$ $5(3) + 2(-2) + 2(1) = 13$ $15 - 4 + 2 = 13$ True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

Note:
Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example:
Exercise:

Problem:
Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:
Equation:

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ -x + 3y - z &= -6 \quad (2) \\ 2x - 5y + 5z &= 17 \quad (3) \end{aligned}$$

Solution:

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

Equation:

$$\begin{array}{rl} x - 2y + 3z = 9 & (1) \\ -x + 3y - z = -6 & (2) \\ \hline y + 2z = 3 & (3) \end{array}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

Equation:

$$\begin{array}{rl} -2x + 4y - 6z = -18 & (1) \text{ multiplied by } -2 \\ 2x - 5y + 5z = 17 & (3) \\ \hline -y - z = -1 & (5) \end{array}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

Equation:

$$\begin{array}{rl} y + 2z = 3 & (4) \\ -y - z = -1 & (5) \\ \hline z = 2 & (6) \end{array}$$

Choosing one equation from each new system, we obtain the upper triangular form:

Equation:

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 2z &= 3 \quad (4) \\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

Equation:

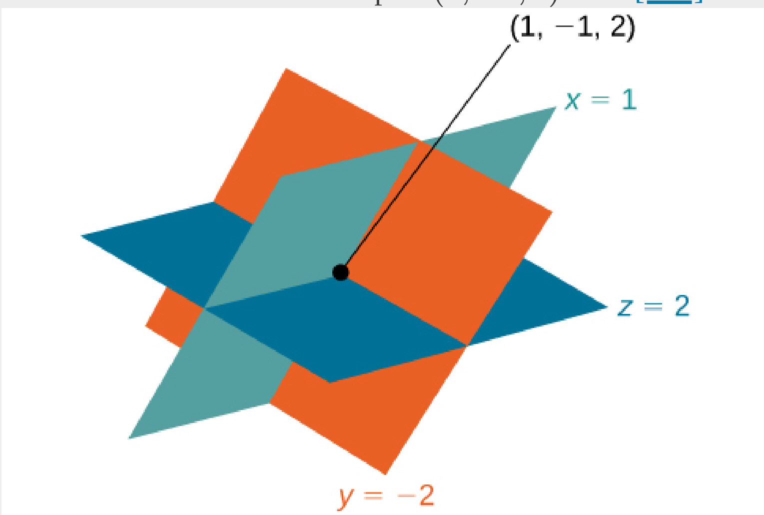
$$y + 2(2) = 3$$
$$y + 4 = 3$$
$$y = -1$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

Equation:

$$x - 2(-1) + 3(2) = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

The solution is the ordered triple $(1, -1, 2)$. See [link].



Example:
Exercise:

Problem:
Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of \$12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested \$4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was \$670. How much did he invest in each type of fund?

Solution:

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:
Equation:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

The first equation indicates that the sum of the three principal amounts is $12,000.
Equation:

$$x + y + z = 12{,}000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.
Equation:

$$z = y + 4{,}000$$

The third equation shows that the total amount of interest earned from each fund equals $670.
Equation:

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.
Equation:

$$\begin{aligned} x + y + z &= 12{,}000 \\ -y + z &= 4{,}000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,
Equation:

$$\begin{aligned} x + \quad y + z \ &= 12{,}000 \quad (1) \\ -y + z \ &= 4{,}000 \quad\ \ (2) \\ 3x + 4y + 7z &= 67{,}000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.
Equation:

$$\begin{aligned} x + \ \ y + \ \ z &= 12{,}000 \\ 3x + 4y\ + 7z &= 67{,}000 \\ -y \ \ + \ \ z &= 4{,}000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.
Equation:

$$\begin{aligned} x + y + z\ &= 12{,}000 \\ y + 4z &= 31{,}000 \\ -y + z\ &= 4{,}000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).
Equation:

$$\begin{aligned} x + y + \ \ z &= 12{,}000 \\ y + 4z &= 31{,}000 \\ 5z\ &= 35{,}000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.
Equation:

$$\begin{aligned} 5z &= 35{,}000 \\ z &= 7{,}000 \end{aligned}$$

$$\begin{aligned} y + 4(7{,}000) &= 31{,}000 \\ y &= 3{,}000 \end{aligned}$$

$$\begin{aligned} x + 3{,}000 + 7{,}000 &= 12{,}000 \\ x &= 2{,}000 \end{aligned}$$

John invested $2,000 in a money-market fund, $3,000 in municipal bonds, and $7,000 in mutual funds.

Note:
Exercise:

Problem: Solve the system of equations in three variables.
Equation:

$$\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned}$$

Solution:

$(1, -1, 1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.
Equation:

$$\begin{aligned} x - 3y + z &= 4 & (1) \\ -x + 2y - 5z &= 3 & (2) \\ 5x - 13y + 13z &= 8 & (3) \end{aligned}$$

Solution:

Looking at the coefficients of $x,$ we can see that we can eliminate x by adding equation (1) to equation (2).
Equation:

$$\begin{array}{rl} x - 3y + z = 4 & (1) \\ -x + 2y - 5z = 3 & (2) \\ \hline -y - 4z = 7 & (4) \end{array}$$

Next, we multiply equation (1) by -5 and add it to equation (3).
Equation:

$$\begin{array}{ll} -5x + 15y - 5z = -20 & (1)\ \text{multiplied by} -5 \\ 5x - 13y + 13z = 8 & (3) \\ \hline 2y + 8z = -12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).
Equation:

$$\begin{array}{ll} -2y - 8z = 14 & (4)\ \text{multiplied by}\ 2 \\ 2y + 8z = -12 & (5) \\ \hline 0 = 2 & \end{array}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Note:
Exercise:

Problem: Solve the system of three equations in three variables.
Equation:

$$\begin{array}{r} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{array}$$

Solution:

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Example:
Exercise:

Problem:
Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.
Equation:

$$\begin{aligned} 2x + y - 3z &= 0 \quad (1) \\ 4x + 2y - 6z &= 0 \quad (2) \\ x - y + z &= 0 \quad (3) \end{aligned}$$

Solution:

First, we can multiply equation (1) by -2 and add it to equation (2).
Equation:

$$\begin{array}{rl} -4x - 2y + 6z = 0 & \text{equation } (1) \text{ multiplied by } -2 \\ 4x + 2y - 6z = 0 & \qquad\qquad (2) \\ \hline 0 = 0 & \end{array}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$,which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2,and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have
Equation:

$$\begin{aligned} 2x + y - 3z &= 0 \\ x - y + z &= 0 \\ \hline 3x - 2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

Equation:

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

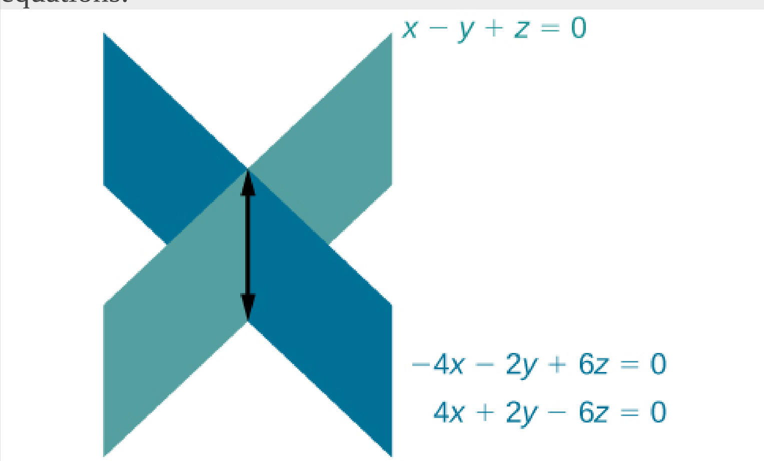
Equation:

$$\begin{aligned} 2x + y - 3\left(\frac{3}{2}x\right) &= 0 \\ 2x + y - \frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x - 2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in [link], two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.



Note:
Does the generic solution to a dependent system always have to be written in terms of x?
No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Note:
Exercise:

Problem: Solve the following system.
Equation:

$$\begin{aligned} x + y + z &= 7 \\ 3x - 2y - z &= 4 \\ x + 6y + 5z &= 24 \end{aligned}$$

Solution:

Infinite number of solutions of the form $(x, 4x - 11, -5x + 18)$.

Note:
Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple $\{(x, y, z)\}$ that represents the intersection of three planes in space. See [link].
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See [link].
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See [link].
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See [link].

- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See [link].
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Section Exercises

Verbal

Exercise:

Problem:

Can a linear system of three equations have exactly two solutions? Explain why or why not

---

Solution:

No, there can be only one, zero, or infinitely many solutions.

Exercise:

Problem:

If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

Exercise:

Problem:

If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Solution:

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x+3y-6z&=1\\ -4x-6y+12z&=-2\\ x+2y+5z&=10 \end{aligned}$$

Exercise:

Problem: Using the method of addition, is there only one way to solve the system?

Exercise:

Problem:

Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Solution:

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

Exercise:

Problem: $\begin{array}{r} 2x-6y+6z=-12 \\ x+4y+5z=-1 \\ -\mathrm{x}+2y+3z=-1 \end{array}$ and $(0, 1, -1)$

Exercise:

Problem: $\begin{array}{r} 6x-y+3z=6 \\ 3x+5y+2z=0 \\ x+y=0 \end{array}$ and $(3, -3, -5)$

Solution:

No

Exercise:

Problem: $\begin{array}{r} 6x-7y+z=2 \\ -\mathrm{x}-y+3z=4 \\ 2x+y-z=1 \end{array}$ and $(4, 2, -6)$

Exercise:

Problem: $\begin{array}{r} x-y=0 \\ x-z=5 \\ x-y+z=-1 \end{array}$ and $(4, 4, -1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{c} -\mathrm{x}-y+2z=3 \\ 5x+8y{-}3z=4 \\ -\mathrm{x}+3y{-}5z=-5 \end{array}$ and $(4,1,-7)$

For the following exercises, solve each system by substitution.

Exercise:

Problem: $\begin{array}{c} 3x{-}4y+2z=-15 \\ 2x+4y+z=16 \\ 2x+3y+5z=20 \end{array}$

Solution:

$(-1,4,2)$

Exercise:

Problem: $\begin{array}{c} 5x{-}2y+3z=20 \\ 2x{-}4y{-}3z=-9 \\ x+6y{-}8z=21 \end{array}$

Exercise:

Problem: $\begin{array}{c} 5x+2y+4z=9 \\ -3x+2y+z=10 \\ 4x{-}3y+5z=-3 \end{array}$

Solution:

$\left(-\frac{85}{107},\frac{312}{107},\frac{191}{107}\right)$

Exercise:

Problem: $\begin{array}{c} 4x{-}3y+5z=31 \\ -x+2y+4z=20 \\ x+5y{-}2z=-29 \end{array}$

Exercise:

Problem:
$$\begin{aligned} 5x-2y+3z&=4\\ -4x+6y-7z&=-1\\ 3x+2y-z&=4 \end{aligned}$$

Solution:

$\left(1,\frac{1}{2},0\right)$

Exercise:

Problem:
$$\begin{aligned} 4x+6y+9z&=0\\ -5x+2y-6z&=3\\ 7x-4y+3z&=-3 \end{aligned}$$

For the following exercises, solve each system by Gaussian elimination.

Exercise:

Problem:
$$\begin{aligned} 2x-y+3z&=17\\ -5x+4y-2z&=-46\\ 2y+5z&=-7 \end{aligned}$$

Solution:

$(4,-6,1)$

Exercise:

Problem:
$$\begin{aligned} 5x-6y+3z&=50\\ -x+4y&=10\\ 2x-z&=10 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 2x+3y-6z&=1\\ -4x-6y+12z&=-2\\ x+2y+5z&=10 \end{aligned}$$

Solution:

$\left(x,\frac{1}{27}(65-16x),\frac{x+28}{27}\right)$

Exercise:

Problem: $$\begin{array}{r} 4x+6y-2z=8 \\ 6x+9y-3z=12 \\ -2x-3y+z=-4 \end{array}$$

Exercise:

Problem: $$\begin{array}{r} 2x+3y-4z=5 \\ -3x+2y+z=11 \\ -x+5y+3z=4 \end{array}$$

Solution:

$\left(-\frac{45}{13},\frac{17}{13},-2\right)$

Exercise:

Problem: $$\begin{array}{r} 10x+2y-14z=8 \\ -\text{x}-2y-4z=-1 \\ -12x-6y+6z=-12 \end{array}$$

Exercise:

Problem: $$\begin{array}{r} x+y+z=14 \\ 2y+3z=-14 \\ -16y-24z=-112 \end{array}$$

Solution:

No solutions exist

Exercise:

Problem: $$\begin{array}{r} 5x-3y+4z=-1 \\ -4x+2y-3z=0 \\ -\text{x}+5y+7z=-11 \end{array}$$

Exercise:

Problem: $$\begin{array}{r} x+y+z=0 \\ 2x-y+3z=0 \\ x-z=0 \end{array}$$

Solution:

$(0, 0, 0)$

Exercise:

Problem: $$\begin{array}{r} 3x + 2y - 5z = 6 \\ 5x - 4y + 3z = -12 \\ 4x + 5y - 2z = 15 \end{array}$$

Exercise:

Problem: $$\begin{array}{r} x + y + z = 0 \\ 2x - y + 3z = 0 \\ x - z = 1 \end{array}$$

Solution:

$\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

Exercise:

Problem: $$\begin{array}{c} 3x - \frac{1}{2}y - z = -\frac{1}{2} \\ 4x + z = 3 \\ -x + \frac{3}{2}y = \frac{5}{2} \end{array}$$

Exercise:

Problem: $$\begin{array}{c} 6x - 5y + 6z = 38 \\ \frac{1}{5}x - \frac{1}{2}y + \frac{3}{5}z = 1 \\ -4x - \frac{3}{2}y - z = -74 \end{array}$$

Solution:

$(7, 20, 16)$

Exercise:

Problem: $$\begin{array}{c} \frac{1}{2}x - \frac{1}{5}y + \frac{2}{5}z = -\frac{13}{10} \\ \frac{1}{4}x - \frac{2}{5}y - \frac{1}{5}z = -\frac{7}{20} \\ -\frac{1}{2}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{5}{4} \end{array}$$

Exercise:

Problem: $$\begin{array}{l} -\frac{1}{3}x - \frac{1}{2}y - \frac{1}{4}z = \frac{3}{4} \\ -\frac{1}{2}x - \frac{1}{4}y - \frac{1}{2}z = 2 \\ -\frac{1}{4}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{1}{2} \end{array}$$

Solution:

$(-6, 2, 1)$

Exercise:

Problem: $$\begin{array}{l} \frac{1}{2}x - \frac{1}{4}y + \frac{3}{4}z = 0 \\ \frac{1}{4}x - \frac{1}{10}y + \frac{2}{5}z = -2 \\ \frac{1}{8}x + \frac{1}{5}y - \frac{1}{8}z = 2 \end{array}$$

Exercise:

Problem: $$\begin{array}{r} \frac{4}{5}x - \frac{7}{8}y + \frac{1}{2}z = 1 \\ -\frac{4}{5}x - \frac{3}{4}y + \frac{1}{3}z = -8 \\ -\frac{2}{5}x - \frac{7}{8}y + \frac{1}{2}z = -5 \end{array}$$

Solution:

$(5, 12, 15)$

Exercise:

Problem: $$\begin{array}{l} -\frac{1}{3}x - \frac{1}{8}y + \frac{1}{6}z = -\frac{4}{3} \\ -\frac{2}{3}x - \frac{7}{8}y + \frac{1}{3}z = -\frac{23}{3} \\ -\frac{1}{3}x - \frac{5}{8}y + \frac{5}{6}z = 0 \end{array}$$

Exercise:

Problem: $$\begin{array}{l} -\frac{1}{4}x - \frac{5}{4}y + \frac{5}{2}z = -5 \\ -\frac{1}{2}x - \frac{5}{3}y + \frac{5}{4}z = \frac{55}{12} \\ -\frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z = \frac{5}{3} \end{array}$$

Solution:

$(-5, -5, -5)$

Exercise:

Problem: $$\begin{array}{r}\frac{1}{40}x+\frac{1}{60}y+\frac{1}{80}z=\frac{1}{100}\\ -\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z=-\frac{1}{5}\\ \frac{3}{8}x+\frac{3}{12}y+\frac{3}{16}z=\frac{3}{20}\end{array}$$

Exercise:

Problem: $$\begin{array}{l}0.1x-0.2y+0.3z=2\\ 0.5x-0.1y+0.4z=8\\ 0.7x-0.2y+0.3z=8\end{array}$$

Solution:

$(10, 10, 10)$

Exercise:

Problem: $$\begin{array}{l}0.2x+0.1y-0.3z=0.2\\ 0.8x+0.4y-1.2z=0.1\\ 1.6x+0.8y-2.4z=0.2\end{array}$$

Exercise:

Problem: $$\begin{array}{l}1.1x+0.7y-3.1z=-1.79\\ 2.1x+0.5y-1.6z=-0.13\\ 0.5x+0.4y-0.5z=-0.07\end{array}$$

Solution:

$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

Exercise:

Problem: $$\begin{array}{l}0.5x-0.5y+0.5z=10\\ 0.2x-0.2y+0.2z=4\\ 0.1x-0.1y+0.1z=2\end{array}$$

Exercise:

Problem: $$\begin{array}{l}0.1x+0.2y+0.3z=0.37\\ 0.1x-0.2y-0.3z=-0.27\\ 0.5x-0.1y-0.3z=-0.03\end{array}$$

Solution:

$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

Exercise:

Problem: $$\begin{array}{l} 0.5x - 0.5y - 0.3z = 0.13 \\ 0.4x - 0.1y - 0.3z = 0.11 \\ 0.2x - 0.8y - 0.9z = -0.32 \end{array}$$

Exercise:

Problem: $$\begin{array}{l} 0.5x + 0.2y - 0.3z = 1 \\ 0.4x - 0.6y + 0.7z = 0.8 \\ 0.3x - 0.1y - 0.9z = 0.6 \end{array}$$

Solution:

$(2, 0, 0)$

Exercise:

Problem: $$\begin{array}{l} 0.3x + 0.3y + 0.5z = 0.6 \\ 0.4x + 0.4y + 0.4z = 1.8 \\ 0.4x + 0.2y + 0.1z = 1.6 \end{array}$$

Exercise:

Problem: $$\begin{array}{l} 0.8x + 0.8y + 0.8z = 2.4 \\ 0.3x - 0.5y + 0.2z = 0 \\ 0.1x + 0.2y + 0.3z = 0.6 \end{array}$$

Solution:

$(1, 1, 1)$

Extensions

For the following exercises, solve the system for x, y, and z.

Exercise:

Problem:
$$\begin{aligned} x+y+z &= 3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2} &= 0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3} &= \frac{2}{3} \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 5x-3y-\frac{z+1}{2} &= \frac{1}{2} \\ 6x+\frac{y-9}{2}+2z &= -3 \\ \frac{x+8}{2}-4y+z &= 4 \end{aligned}$$

Solution:

$\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$

Exercise:

Problem:
$$\begin{aligned} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3} &= 1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12} &= 0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2} &= 3 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3} &= 2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2} &= 1 \\ \frac{x+6}{2}-\frac{y-3}{2}+z+1 &= 9 \end{aligned}$$

Solution:

$(6, -1, 0)$

Exercise:

Problem:
$$\begin{aligned} \frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6} &= 1 \\ 4x+3y-2z &= 11 \\ 0.02x+0.015y-0.01z &= 0.065 \end{aligned}$$

Real-World Applications

Exercise:

Problem:

Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

Solution:

24, 36, 48

Exercise:

Problem:

Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

Exercise:

Problem:

At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

Solution:

70 grandparents, 140 parents, 190 children

Exercise:

Problem:

An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

Exercise:

Problem:

Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate’s groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

Solution:

Your share was $19.95, Sarah’s share was $40, and your other roommate’s share was $22.05.

Exercise:

Problem:

Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

Exercise:

Problem:

Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

Solution:

There are infinitely many solutions; we need more information

Exercise:

Problem:

At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

Exercise:

Problem:

A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

Solution:

500 students, 225 children, and 450 adults

Exercise:

Problem:

In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

Exercise:

Problem:

Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

Solution:

The BMW was $49,636, the Jeep was $42,636, and the Toyota was $47,727.

Exercise:

Problem:

A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

Exercise:

Problem:

You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

Solution:

$400,000 in the account that pays 3% interest, $500,000 in the account that pays 4% interest, and $100,000 in the account that pays 2% interest.

Exercise:

Problem:

You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

Exercise:

Problem:

The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil.

Exercise:

Problem:

The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Exercise:

Problem:

The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

Exercise:

Problem:

The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[footnote]

"USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

Exercise:

Problem:

At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Solution:

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

Exercise:

Problem:

Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[footnote]

"The United States Meat Industry at a Glance," accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

Glossary

solution set
: the set of all ordered pairs or triples that satisfy all equations in a system of equations

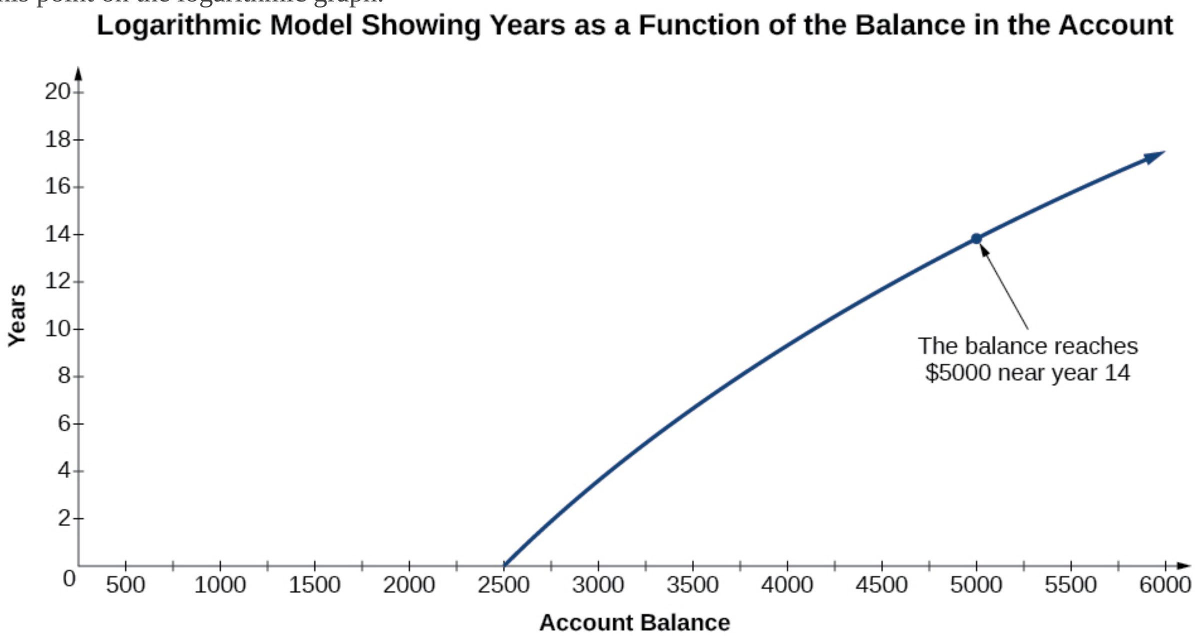
Derived copy of Graphs of Logarithmic Functions
In this section, you will:

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%,compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A=2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? [link] shows this point on the logarithmic graph.



In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y=b^x$ for any real number x and constant $b>0$, $b\neq 1$, where

- The domain of y is $(-\infty,\infty)$.

- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x :

Equation:

$$\begin{array}{rl} 2x - 3 > 0 & \text{Show the argument greater than zero.} \\ 2x > 3 & \text{Add 3.} \\ x > 1.5 & \text{Divide by 2.} \end{array}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Note:
Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

Equation:

$$
\begin{array}{rl}
x+3>0 & \text{The input must be positive.} \\
x>-3 & \text{Subtract 3.}
\end{array}
$$

The domain of $f(x)=\log_2(x+3)$ is $(-3,\infty)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log_5(x-2)+1$?

Solution:

$(2,\infty)$

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x)=\log(5-2x)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x>0.$ Solving this inequality,

Equation:

$$
\begin{array}{rl}
5-2x>0 & \text{The input must be positive.} \\
-2x>-5 & \text{Subtract 5.} \\
x<\frac{5}{2} & \text{Divide by } -2 \text{ and switch the inequality.}
\end{array}
$$

The domain of $f(x)=\log(5-2x)$ is $\left(-\infty,\frac{5}{2}\right)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log(x-5)+2$?

Solution:

$(5,\infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

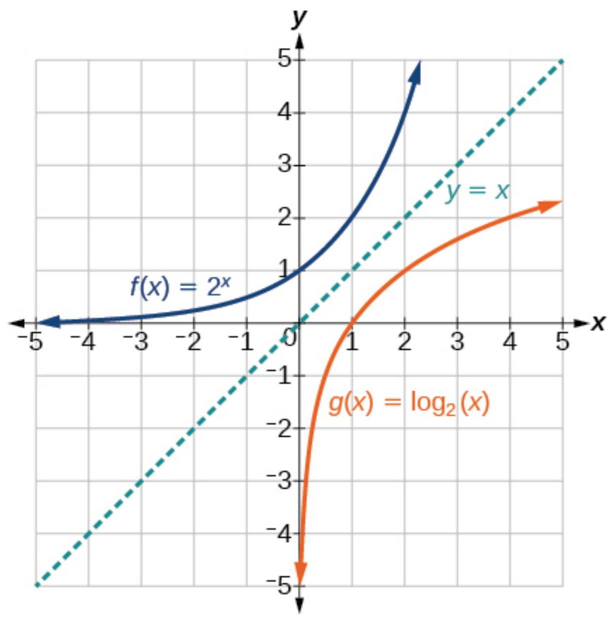
We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in [link].

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from [link], we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See [link].

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the *x*- and *y*-coordinates are reversed for the inverse functions. [link] shows the graph of f and g.



Notice that the graphs of $f\left(x\right)={2}^{x}$ and $g\left(x\right)={\text{log}}_{2}\left(x\right)$ are reflections about the line $y=x$.

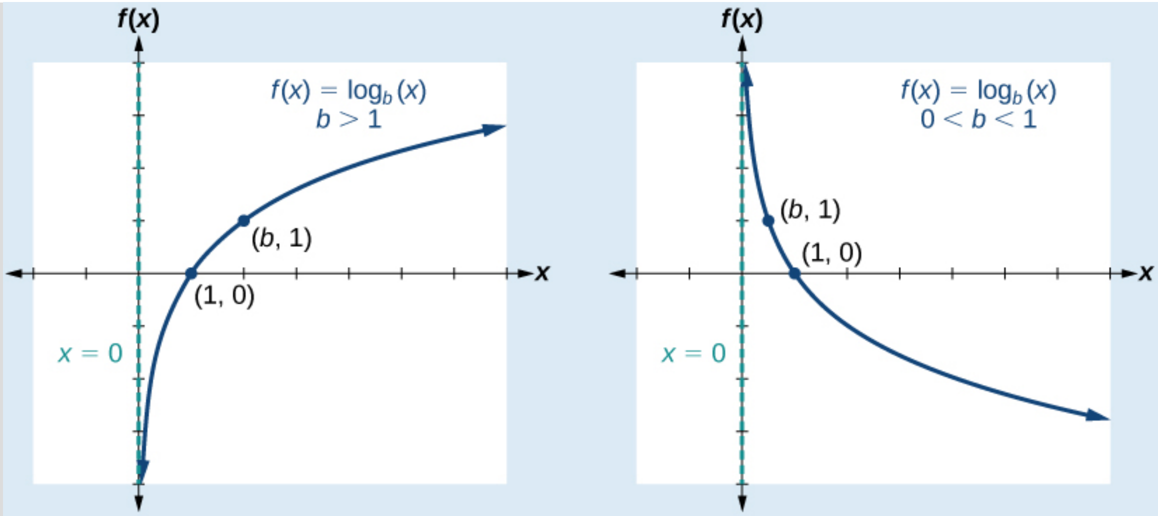
Observe the following from the graph:

- $f(x)={2}^{x}$ has a y-intercept at $(0,1)$ and $g(x)={\text{log}}_{2}\left(x\right)$ has an x- intercept at $(1,0)$.
- The domain of $f(x)={2}^{x}$, $(-\infty ,\infty)$, is the same as the range of $g(x)={\text{log}}_{2}\left(x\right)$.
- The range of $f(x)={2}^{x}$, $(0,\infty)$, is the same as the domain of $g(x)={\text{log}}_{2}\left(x\right)$.

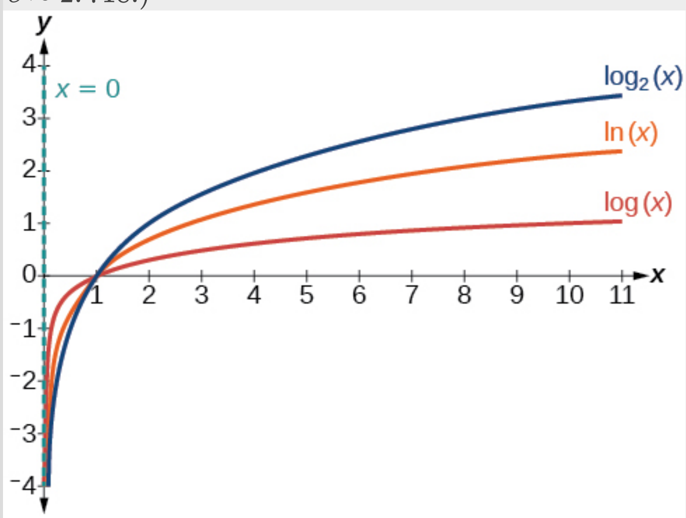
Note:
Characteristics of the Graph of the Parent Function, *f*(*x*) = log$_b$(*x*)
For any real number x and constant $b>0$,$b\ne 1$, we can see the following characteristics in the graph of $f(x)={\text{log}}_{b}\left(x\right)$:

- one-to-one function
- vertical asymptote: $x=0$
- domain: $(0,\infty)$
- range: $(-\infty ,\infty)$
- x-intercept: $(1,0)$ and key point $(b,1)$
- y-intercept: none
- increasing if $b>1$
- decreasing if $0<b<1$

See [link].



[link] shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$.)



The graphs of three logarithmic functions with different bases, all greater than 1.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$,and the vertical asymptote, $x = 0$.

Example:
Exercise:

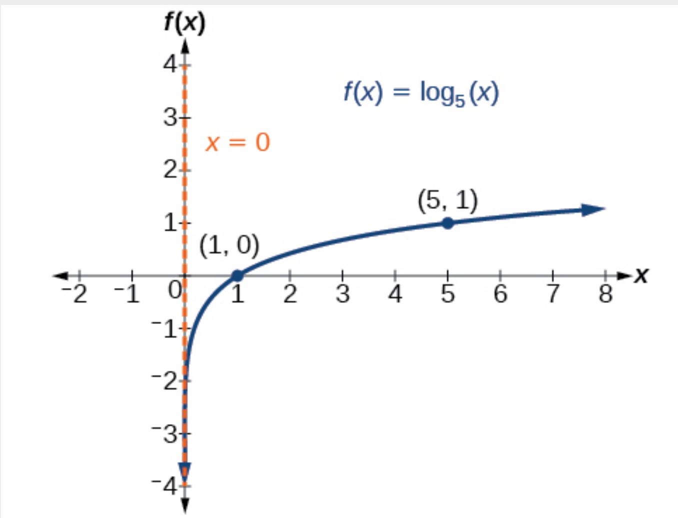
Problem:
Graphing a Logarithmic Function with the Form *f*(*x*) = log*b*(*x*).

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution:

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The *x*-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see [link]).

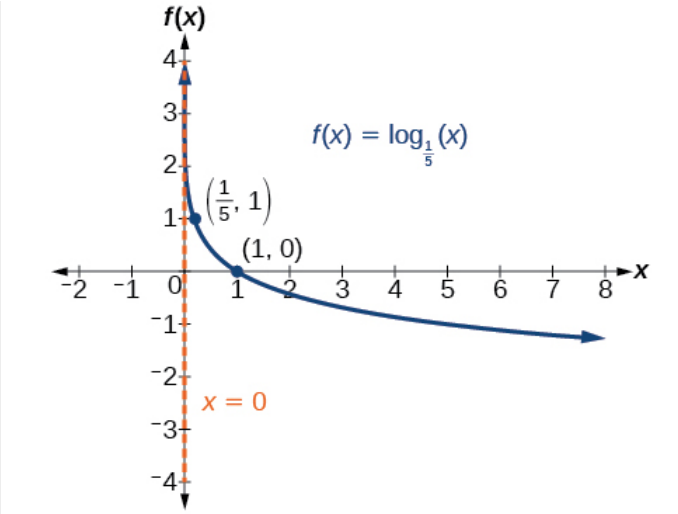


The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem: Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Solution:



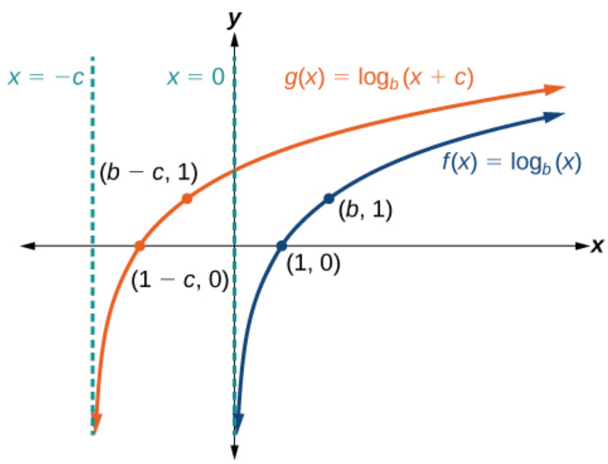
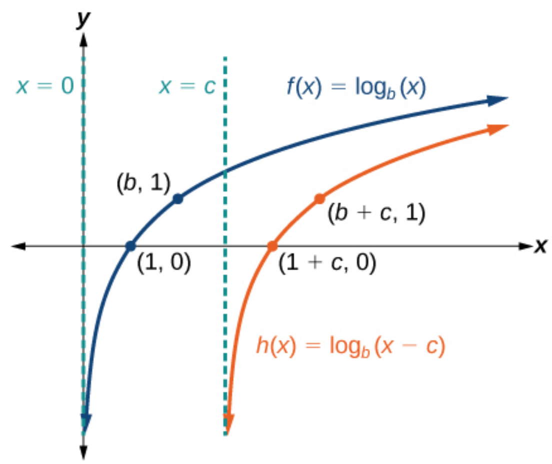
The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See [link].

Shift left $g(x) = \log_b(x + c)$	Shift right $h(x) = \log_b(x - c)$
 •The asymptote changes to $x = -c$. •The domain changes to $(-c, \infty)$. •The range remains $(-\infty, \infty)$.	 •The asymptote changes to $x = c$. •The domain changes to (c, ∞). •The range remains $(-\infty, \infty)$.

Note:
Horizontal Shifts of the Parent Function $y = \log_b(x)$
For any constant c,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:

   a. If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
   b. If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right c units.

2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example:
Exercise:

Problem:
Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$,so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

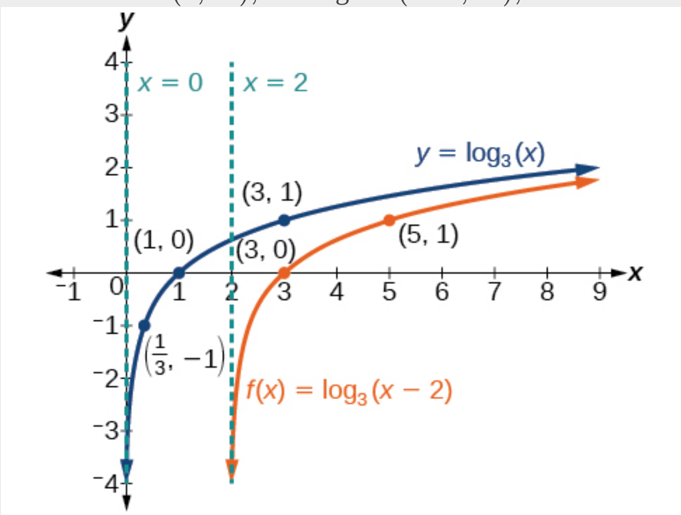
The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$,$(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$,$(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

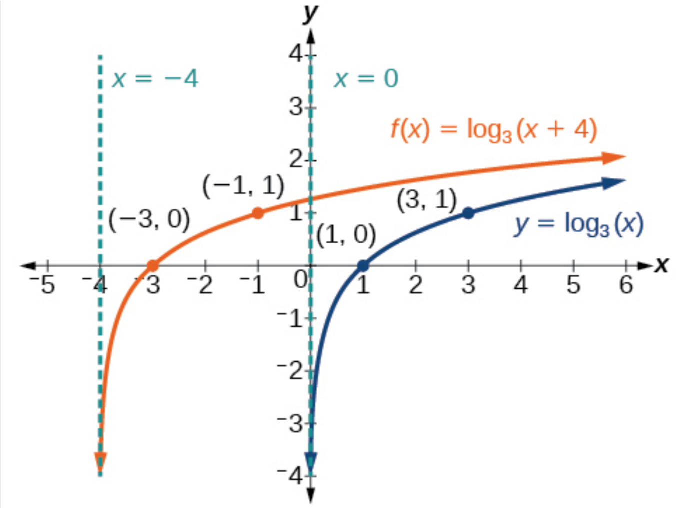


Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

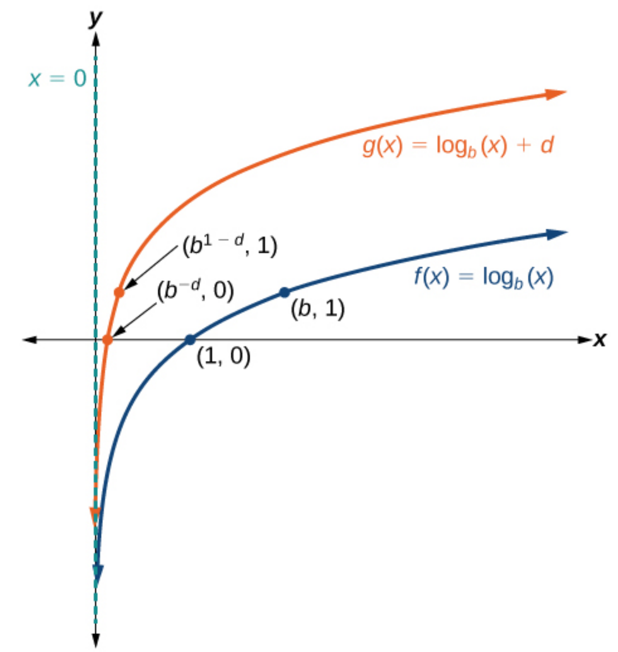
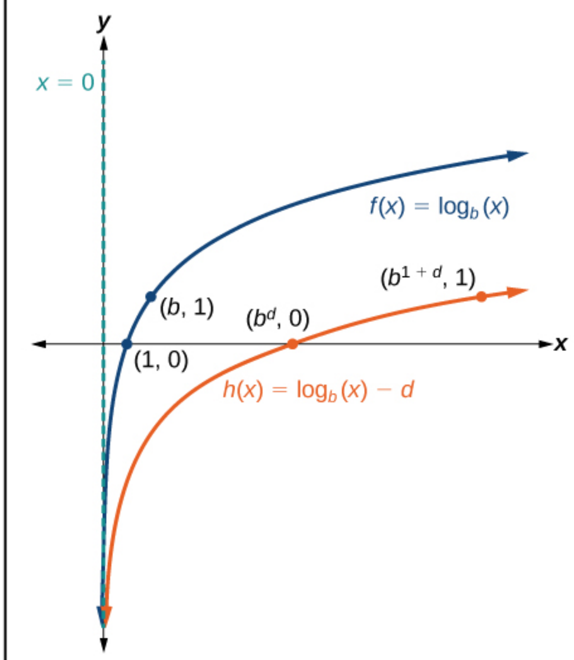
Solution:



The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift of *y* = log*b*(*x*)

When a constant d is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See [link].

Shift up $g(x) = \log_b(x) + d$	**Shift down** $h(x) = \log_b(x) - d$
	
•The asymptote remains $x = 0$. •The domain remains to $(0, \infty)$. •The range remains $(-\infty, \infty)$.	•The asymptote remains $x = 0$. •The domain remains to $(0, \infty)$. •The range remains $(-\infty, \infty)$.

Note:
Vertical Shifts of the Parent Function $y = \log_b(x)$
For any constant d,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$down d units.
2. Draw the vertical asymptote $x = 0$.

3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
4. Label the three points.
5. The domain is $(0,\infty)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x=0$.

Example:
Exercise:

Problem:
Graphing a Vertical Shift of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f(x)=\log_3(x)-2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x)=\log_3(x)-2$,we will notice $d=-2$. Thus $d<0$.

This means we will shift the function $f(x)=\log_3(x)$ down 2 units.

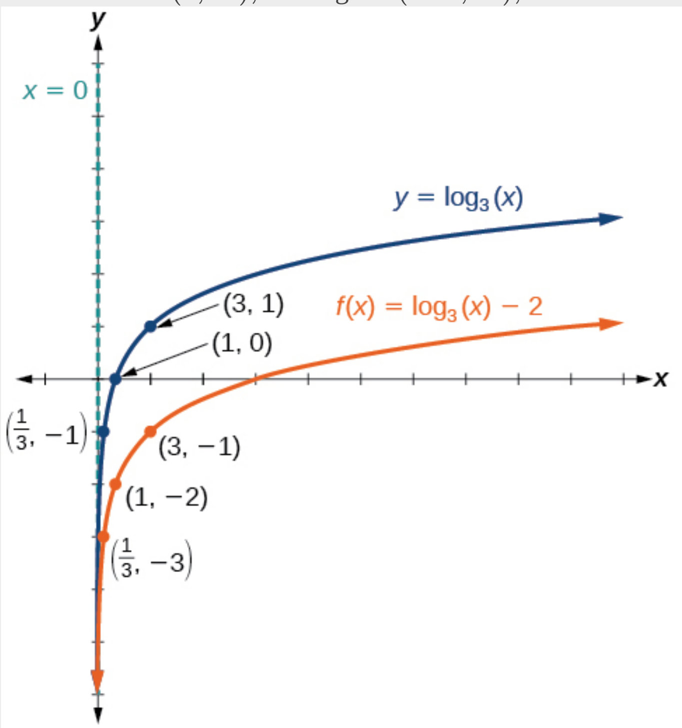
The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{3},-1\right)$,$(1,0)$,and $(3,1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3},-3\right)$,$(1,-2)$, and $(3,-1)$.

The domain is $(0,\infty)$,the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.



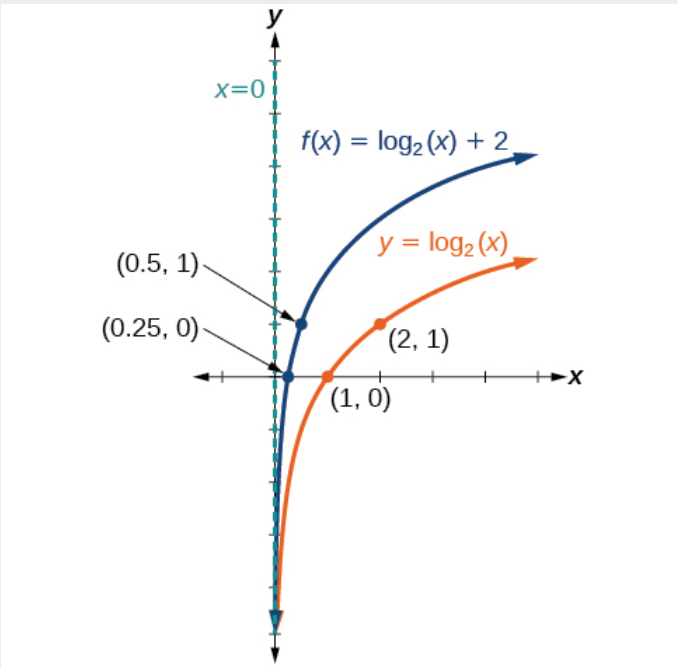
The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

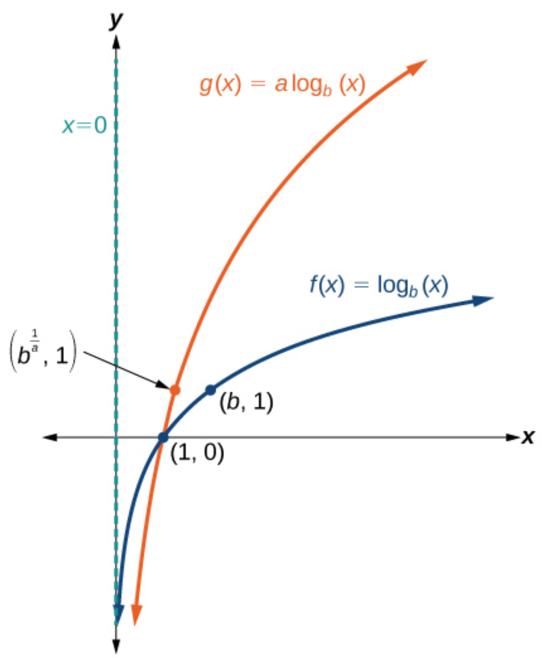
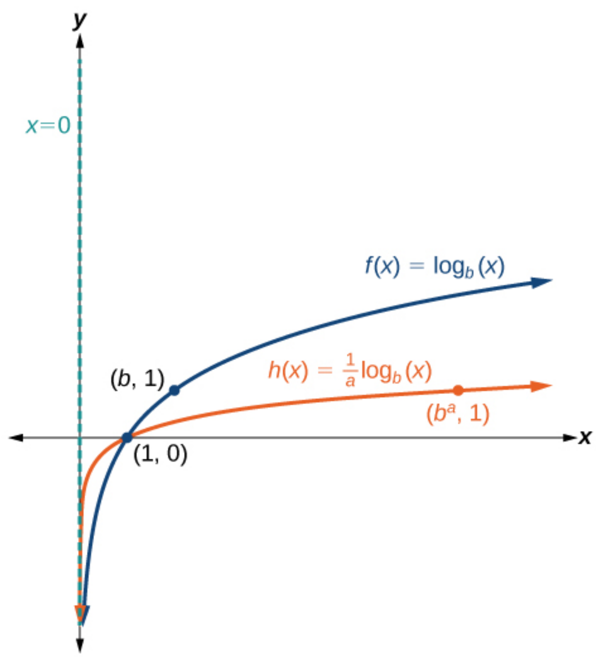
Solution:



The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.See [link].

Vertical Stretch $g(x) = a\log_b(x)$, $a > 1$	Vertical Compression $h(x) = \frac{1}{a}\log_b(x)$, $a > 1$
 •The asymptote remains $x = 0$. •The x-intercept remains (1, 0). •The domain remains $(0, \infty)$. •The range remains $(-\infty, \infty)$.	 •The asymptote remains $x = 0$. •The x-intercept remains (1, 0). •The domain remains $(0, \infty)$. •The range remains $(-\infty, \infty)$.

Note:
Vertical Stretches and Compressions of the Parent Function $y = \log_b(x)$
For any constant $a > 1$,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = a\log_b(x)$,$a > 0$,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$,the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
   - If $|a| < 1$,the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
4. Label the three points.
5. The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example:
Exercise:

Problem:
Graphing a Stretch or Compression of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

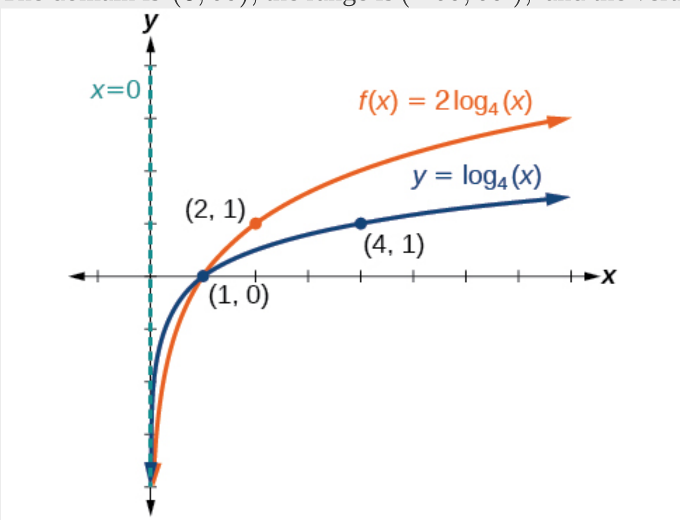
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$,$(1, 0\ )$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$,$(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty\ )$, and the vertical asymptote is $x = 0$. See [link].



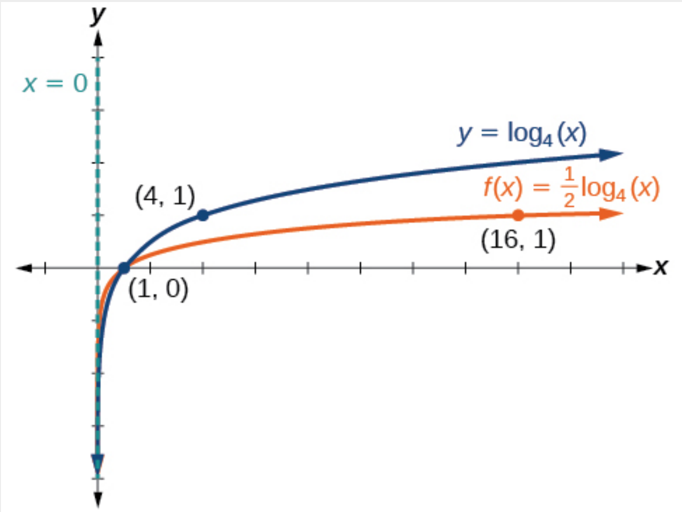
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:



The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

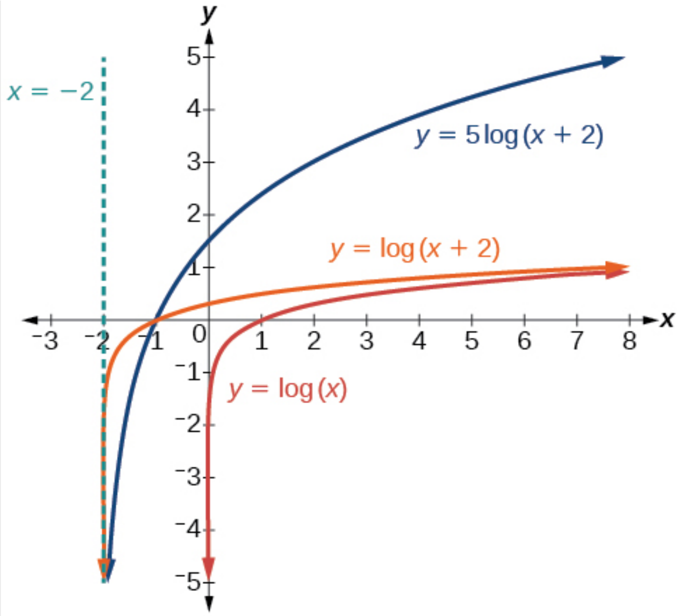
Example:
Exercise:

Problem:
Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution:

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in [link]. The vertical asymptote will be shifted to $x = -2$. The *x*-intercept will be $(-1,0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5\ )$. We chose $x = 8$ as the *x*-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.



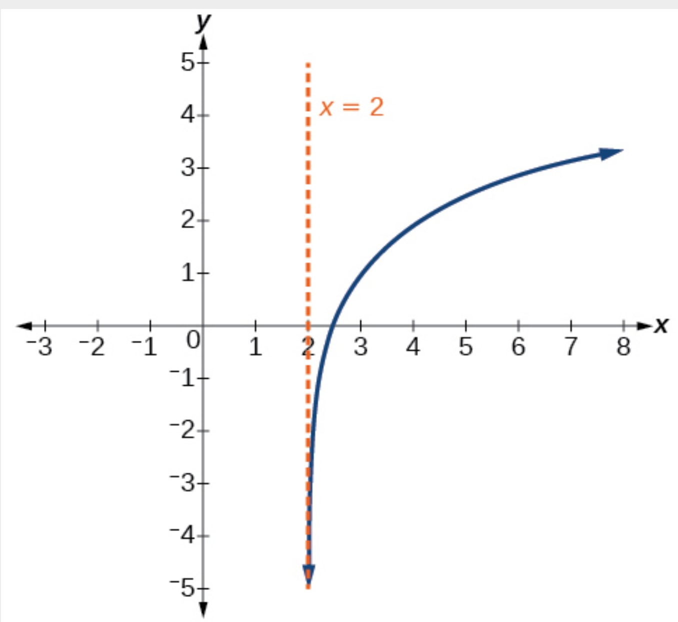
The domain is $(-2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

Note:
Exercise:

Problem:

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

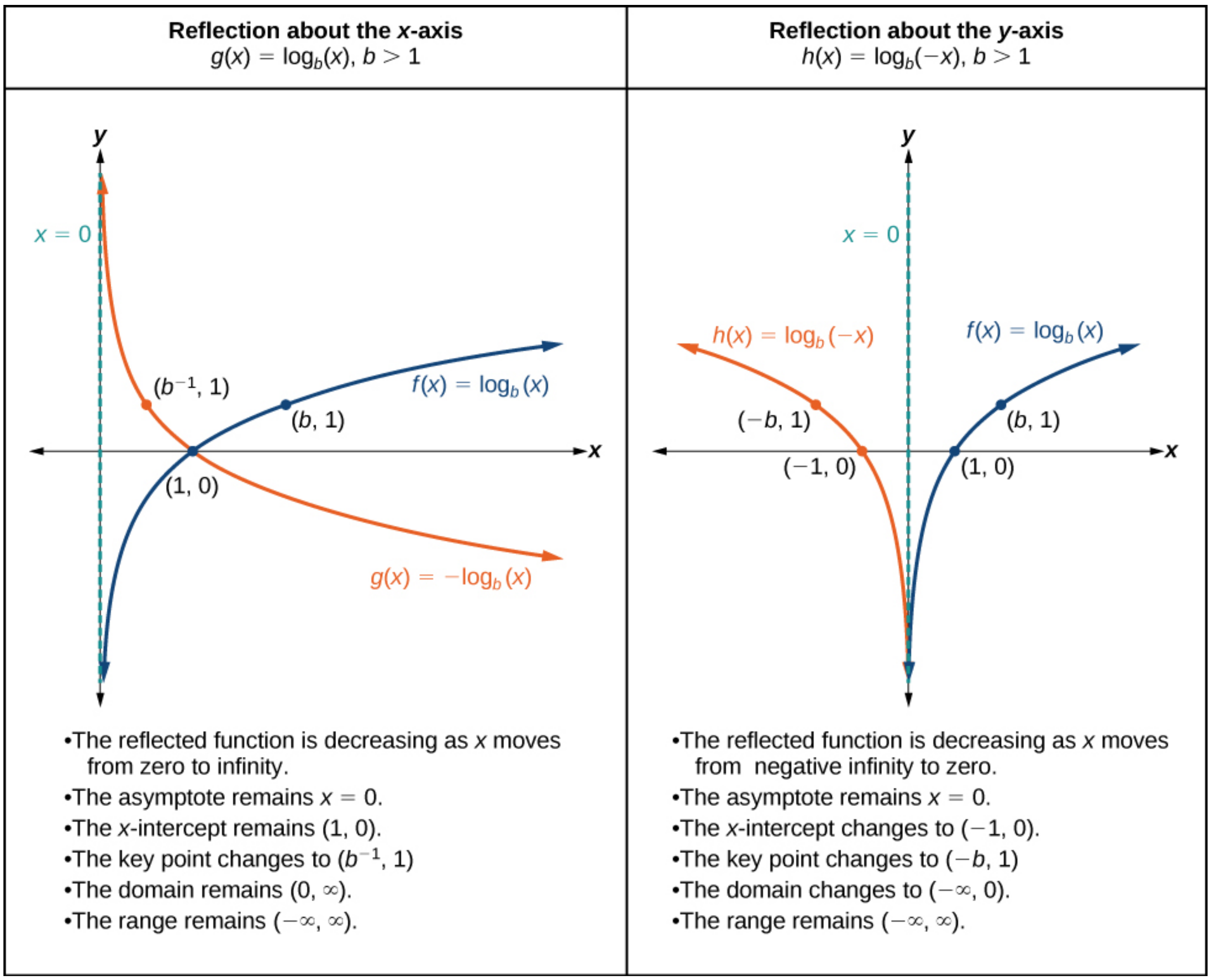
Solution:



The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

Graphing Reflections of *f*(*x*) = log$_b$(*x*)

When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the *x*-axis. When the *input* is multiplied by -1,the result is a reflection about the *y*-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the *x*-axis, $g(x) = -\log_b(x)$ and the reflection about the *y*-axis, $h(x) = \log_b(-x)$.



Note:
Reflections of the Parent Function *y* = log$_b$(*x*)
The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the *x*-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the *y*-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Note:
Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	**If $f(x) = \log_b(-x)$**
(2,1)	(2,2)
(3,1)	(3,2)
(4,1)	(4,2)
(5,1)	(5,2)
(6,1)	(6,2)

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.	2. Plot the *x*-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.

If $f(x)=-\log_b(x)$	If $f(x)=\log_b(-x)$
5. State the domain, $(0,\infty)$, the range, $(-\infty,\infty)$, and the vertical asymptote $x=0$.	5. State the domain, $(-\infty,0)$, the range, $(-\infty,\infty)$, and the vertical asymptote $x=0$.

Example:
Exercise:

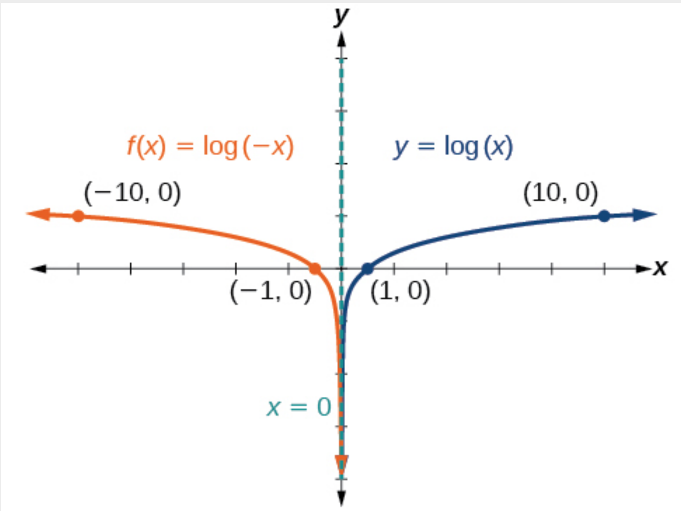
Problem:
Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x)=\log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Before graphing $f(x)=\log(-x)$,identify the behavior and key points for the graph.

- Since $b=10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1,f is a reflection of the parent graph about the *y*-axis. Thus, $f(x)=\log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x=0$.
- The *x*-intercept is $(-1,0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

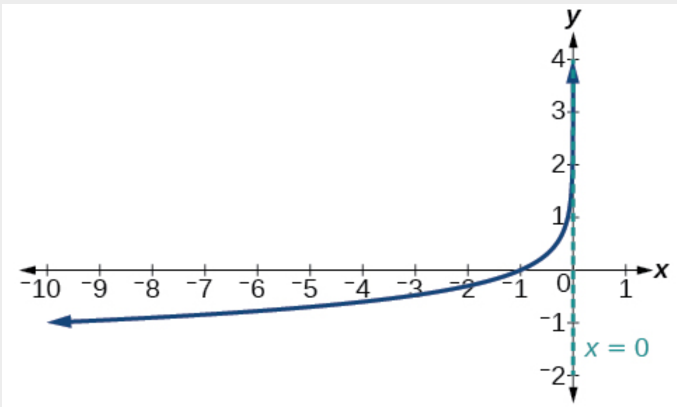


The domain is $(-\infty,0)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x=0$.

Note:
Exercise:

Problem: Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Solution:



The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x,for the point(s) of intersection.

Example:
Exercise:

Problem:
Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x-1)$ graphically. Round to the nearest thousandth.

Solution:

Press **[Y=]** and enter $4\ln(x) + 1$ next to **Y_1=**. Then enter $-2\ln(x-1)$ next to **Y_2=**. For a window, use the values 0 to 5 for x and –10 to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be

different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

Note:
Exercise:

Problem: Solve $5 \log(x+2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

Solution:

$x \approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y = \log_b(x)$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x+c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a\log_b(x)$
Reflect about the *x*-axis	$y = -\log_b(x)$
Reflect about the *y*-axis	$y = \log_b(-x)$
General equation for all translations	$y = a\log_b(x+c) + d$

Note:
Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form
Equation:

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x), b > 1$,is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

Example:
Exercise:

Problem:
Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$?

Solution:

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Note:
Exercise:

Problem: What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

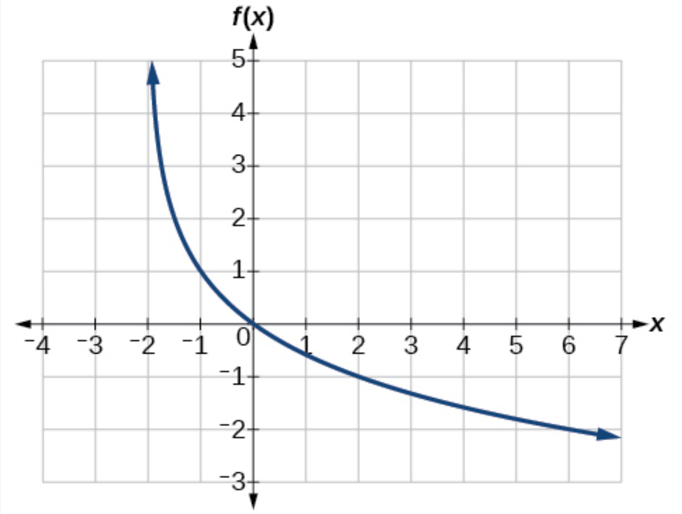
Solution:

$x = 1$

Example:
Exercise:

Problem:
Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in [link].



Solution:

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

Equation:

$$f(x) = -a \log(x+2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

Equation:

$$\begin{array}{ll} 1 = -a \log(-1+2) + k & \text{Substitute } (-1,1). \\ 1 = -a \log(1) + k & \text{Arithmetic.} \\ 1 = k & \log(1) = 0. \end{array}$$

Next, substituting in $(2, -1)$,

Equation:

$$\begin{array}{ll} -1 = -a \log(2+2) + 1 & \text{Plug in } (2,-1). \\ -2 = -a \log(4) & \text{Arithmetic.} \\ a = \frac{2}{\log(4)} & \text{Solve for } a. \end{array}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)} \log(x+2) + 1$.

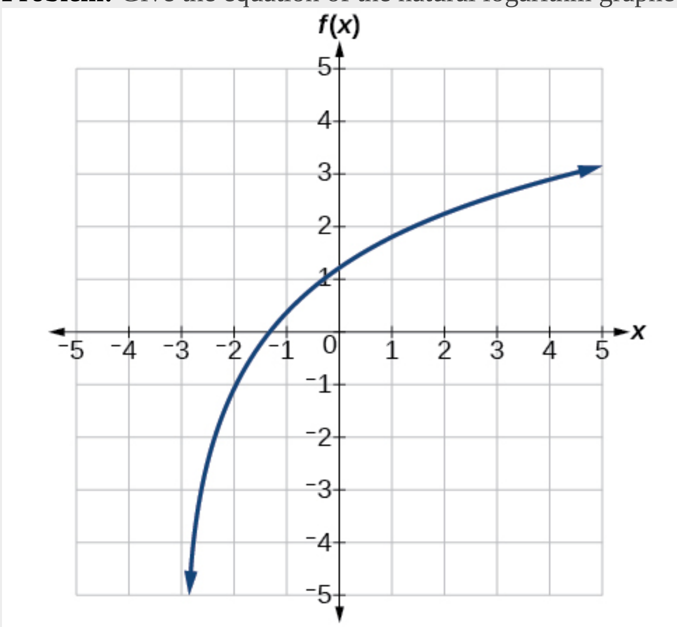
Analysis

We can verify this answer by comparing the function values in [link] with the points on the graph in [link].

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

Note:
Exercise:

Problem: Give the equation of the natural logarithm graphed in [link].



Solution:

$f(x) = 2\ln(x+3) - 1$

Note:
Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?
Yes, if we know the function is a general logarithmic function. For example, look at the graph in [link]. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $\{x \mid x > -3\}$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$.

Note:
Access these online resources for additional instruction and practice with graphing logarithms.

- [Graph an Exponential Function and Logarithmic Function]
- [Match Graphs with Exponential and Logarithmic Functions]
- [Find the Domain of Logarithmic Functions]

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x)=\log_b(x)$	$f(x)=a\log_b(x+c)+d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See [link] and [link]
- The graph of the parent function $f(x)=\log_b(x)$ has an x-intercept at $(1,0)$,domain $(0,\infty)$,range $(-\infty,\infty)$,vertical asymptote $x=0$,and
  - if $b>1$,the function is increasing.
  - if $0<b<1$, the function is decreasing.

  See [link].
- The equation $f(x)=\log_b(x+c)$ shifts the parent function $y=\log_b(x)$ horizontally
  - left c units if $c>0$.
  - right c units if $c<0$.

  See [link].
- The equation $f(x)=\log_b(x)+d$ shifts the parent function $y=\log_b(x)$ vertically
  - up d units if $d>0$.
  - down d units if $d<0$.

  See [link].
- For any constant $a>0$, the equation $f(x)=a\log_b(x)$
  - stretches the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|>1$.
  - compresses the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|<1$.

  See [link] and [link].
- When the parent function $y=\log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
  - The equation $f(x)=-\log_b(x)$ represents a reflection of the parent function about the *x*-axis.

- The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the y-axis.

See [link].

- A graphing calculator may be used to approximate solutions to some logarithmic equations See [link].

- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x+c) + d$. See [link].
- Given an equation with the general form $f(x) = a\log_b(x+c) + d$,we can identify the vertical asymptote $x = -c$ for the transformation. See [link].
- Using the general equation $f(x) = a\log_b(x+c) + d$,we can write the equation of a logarithmic function given its graph. See [link].

Section Exercises

Verbal

Exercise:

Problem:

The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Solution:

Since the functions are inverses, their graphs are mirror images about the line $y = x$. So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

Exercise:

Problem: What type(s) of translation(s), if any, affect the range of a logarithmic function?

Exercise:

Problem: What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Solution:

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

Exercise:

Problem: Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?

Exercise:

Problem: Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Solution:

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

Exercise:

Problem: $f(x) = \log_3(x+4)$

Exercise:

Problem: $h(x) = \ln\left(\frac{1}{2} - x\right)$

Solution:

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $g(x) = \log_5(2x+9) - 2$

Exercise:

Problem: $h(x) = \ln(4x+17) - 5$

Solution:

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

Exercise:

Problem: $f(x) = \log_b(x-5)$

Solution:

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

Exercise:

Problem: $g(x) = \ln(3-x)$

Exercise:

Problem: $f(x) = \log(3x+1)$

Solution:

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

Exercise:

Problem: $f(x) = 3\log(-x) + 2$

Exercise:

Problem: $g(x) = -\ln(3x + 9) - 7$

Solution:

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

Exercise:

Problem: $f(x) = \ln(2 - x)$

Exercise:

Problem: $f(x) = \log\left(x - \frac{3}{7}\right)$

Solution:

Domain: $\left(\frac{3}{7}, \infty\right)$;
Vertical asymptote: $x = \frac{3}{7}$; End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

Exercise:

Problem: $h(x) = -\log(3x - 4) + 3$

Exercise:

Problem: $g(x) = \ln(2x + 6) - 5$

Solution:

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$;
End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

Exercise:

Problem: $f(x) = \log_3(15 - 5x) + 6$

For the following exercises, state the domain, range, and *x*- and *y*-intercepts, if they exist. If they do not exist, write DNE.

Exercise:

Problem: $h(x) = \log_4(x - 1) + 1$

Solution:

Domain: $(1, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 1$; x-intercept: $\left(\frac{5}{4}, 0\right)$; y-intercept: DNE

Exercise:

Problem: $f(x) = \log(5x + 10) + 3$

Exercise:

Problem: $g(x) = \ln(-x) - 2$

Solution:

Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; x-intercept: $\left(-e^2, 0\right)$; y-intercept: DNE

Exercise:

Problem: $f(x) = \log_2(x + 2) - 5$

Exercise:

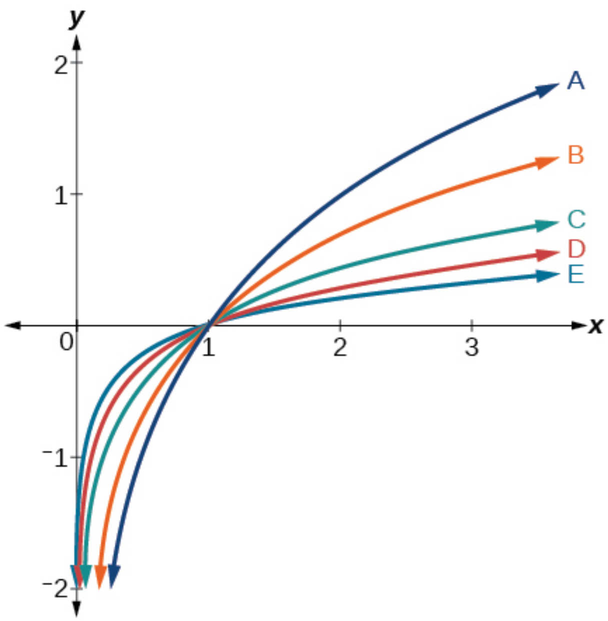
Problem: $h(x) = 3\ln(x) - 9$

Solution:

Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; x-intercept: $\left(e^3, 0\right)$; y-intercept: DNE

Graphical

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $d(x) = \log(x)$

Exercise:

Problem: $f(x) = \ln(x)$

Solution:

B

Exercise:

Problem: $g(x) = \log_2(x)$

Exercise:

Problem: $h(x) = \log_5(x)$

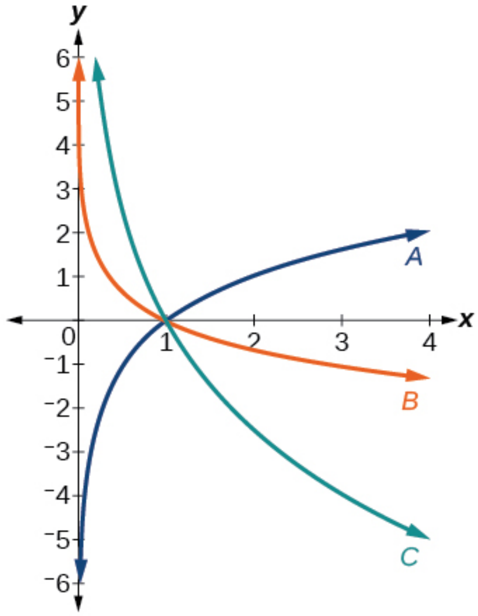
Solution:

C

Exercise:

Problem: $j(x) = \log_{25}(x)$

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $f(x) = \log_{\frac{1}{3}}(x)$

Solution:

B

Exercise:

Problem: $g(x) = \log_2(x)$

Exercise:

Problem: $h(x) = \log_{\frac{3}{4}}(x)$

Solution:

C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

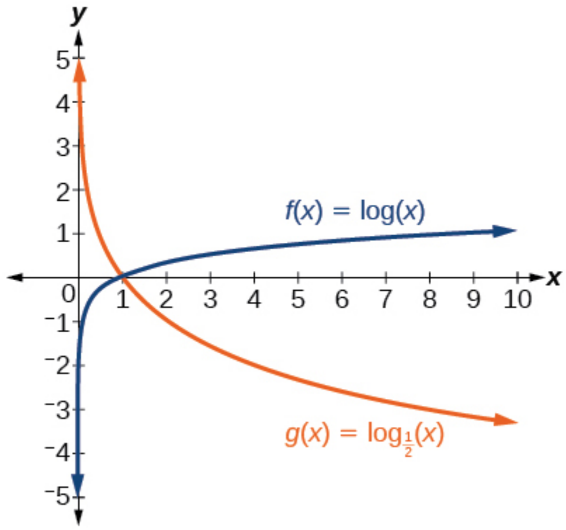
Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = 10^x$

Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

Solution:



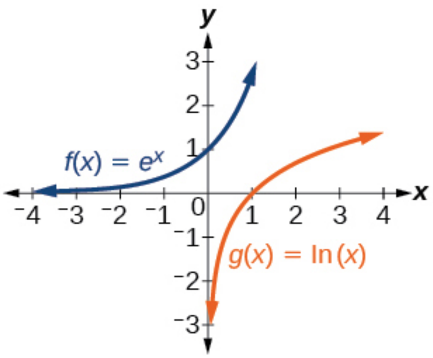
Exercise:

Problem: $f(x) = \log_4(x)$ and $g(x) = \ln(x)$

Exercise:

Problem: $f(x) = e^x$ and $g(x) = \ln(x)$

Solution:



For the following exercises, match each function in [link] with the letter corresponding to its graph.

[missing_resource: CNX_PreCalc_Figure_04_04_207.jpg]

Exercise:

Problem: $f(x) = \log_4(-x+2)$

Exercise:

Problem: $g(x) = -\log_4(x+2)$

Solution:

C

Exercise:

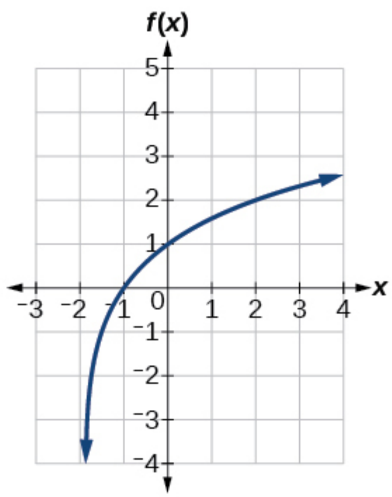
Problem: $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

Exercise:

Problem: $f(x) = \log_2(x+2)$

Solution:



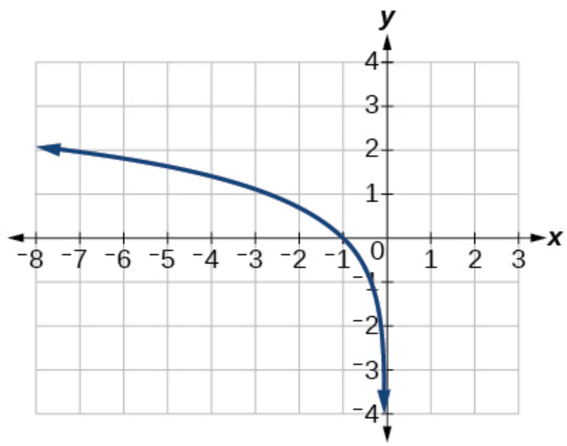
Exercise:

Problem: $f(x) = 2\log(x)$

Exercise:

Problem: $f(x) = \ln(-x)$

Solution:



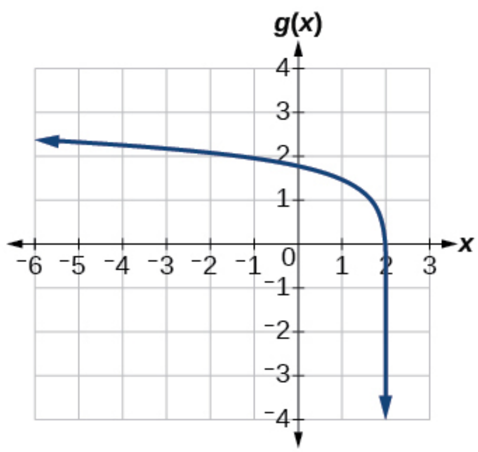
Exercise:

Problem: $g(x) = \log(4x + 16) + 4$

Exercise:

Problem: $g(x) = \log(6 - 3x) + 1$

Solution:



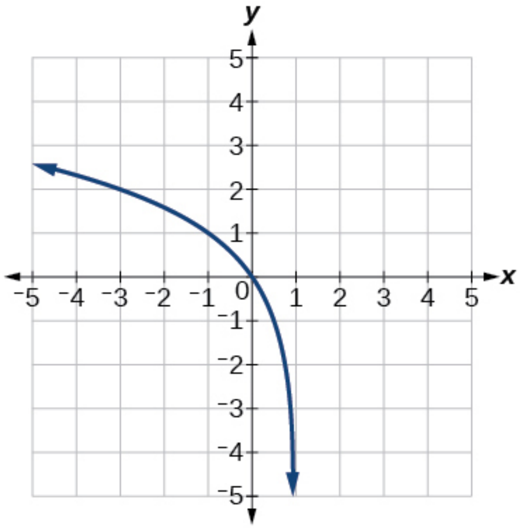
Exercise:

Problem: $h(x) = -\frac{1}{2}\ln(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

Exercise:

Problem: Use $y = \log_2(x)$ as the parent function.

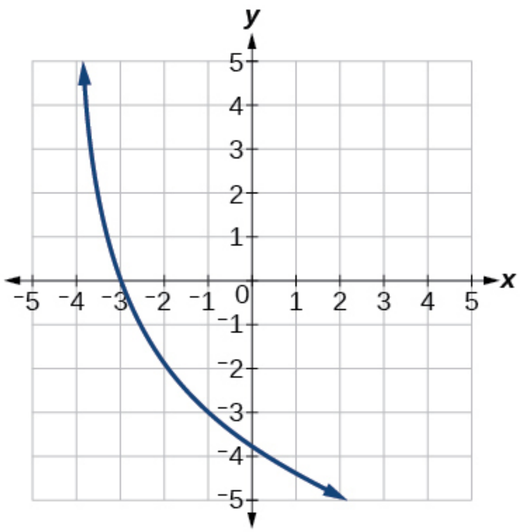


Solution:

$f(x) = \log_2(-(x-1))$

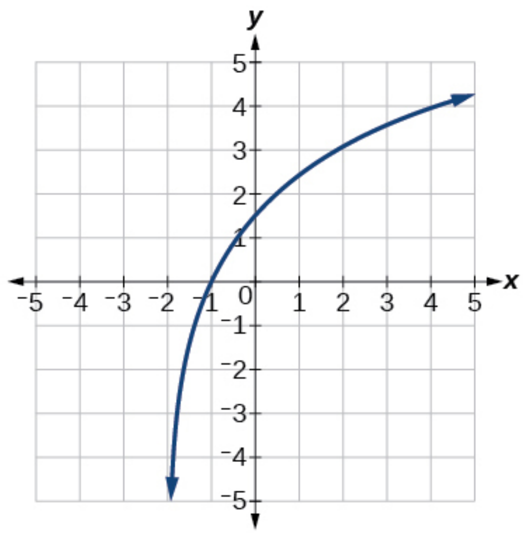
Exercise:

Problem: Use $f(x) = \log_3(x)$ as the parent function.



Exercise:

Problem: Use $f(x) = \log_4(x)$ as the parent function.

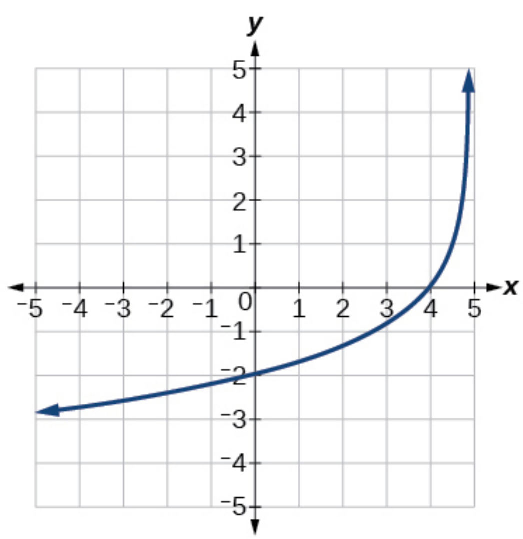


Solution:

$f(x) = 3\log_4(x+2)$

Exercise:

Problem: Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

Exercise:

Problem: $\log(x-1)+2 = \ln(x-1)+2$

Solution:

$x = 2$

Exercise:

Problem: $\log(2x-3)+2=-\log(2x-3)+5$

Exercise:

Problem: $\ln(x-2)=-\ln(x+1)$

Solution:

$x \approx 2.303$

Exercise:

Problem: $2\ln(5x+1)=\frac{1}{2}\ln(-5x)+1$

Exercise:

Problem: $\frac{1}{3}\log(1-x)=\log(x+1)+\frac{1}{3}$

Solution:

$x \approx -0.472$

Extensions

Exercise:

Problem:

Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

Exercise:

Problem:

Explore and discuss the graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$. Make a conjecture based on the result.

Solution:

The graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$ appear to be the same; Conjecture: for any positive base $b \neq 1$, $\log_b(x)=-\log_{\frac{1}{b}}(x)$.

Exercise:

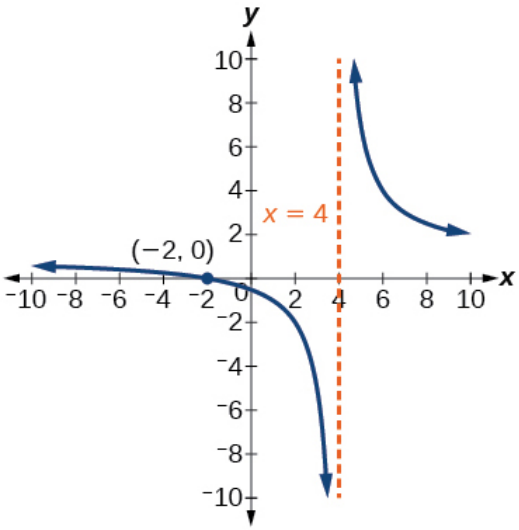
Problem: Prove the conjecture made in the previous exercise.

Exercise:

Problem: What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Solution:

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f\left(x\right) = \frac{x+2}{x-4}$, note that the graph lies above the *x*-axis on the interval $\left(-\infty, -2\right)$ and again to the right of the vertical asymptote, that is $\left(4, \infty\right)$. Therefore, the domain is $\left(-\infty, -2\right) \cup \left(4, \infty\right)$.



Exercise:

Problem:

Use properties of exponents to find the *x*-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.